particular numerical values chosen for $\beta_{1}, \cdots, \beta_{n-l}$, since (6) holds for all $\beta$.

By duality, the results apply to $m$-input single-output systems as

$$
\begin{align*}
& {\left[\begin{array}{c}
k_{1} \\
\vdots \\
k_{m} \\
a_{1} \\
\vdots \\
\cdot \\
a_{n-m}
\end{array}\right]=\left[\begin{array}{cccc}
w\left(\alpha_{1}\right) & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
w\left(\alpha_{m}\right) & 0 & \cdots & 0 \\
w\left(\beta_{1}\right) & -P\left(\beta_{1}\right) \beta_{1}^{n-m-1} & \cdots & -P\left(\beta_{1}\right) \\
\vdots & \vdots & & \vdots \\
\vdots\left(\beta_{n-m}\right) & -P\left(\beta_{n-m}\right) \beta_{n-m}^{n-m-1} & \cdots & -P\left(\beta_{n-m}\right)
\end{array}\right]} \\
& {\left[\begin{array}{c}
-F\left(\alpha_{1}\right) \\
\vdots \\
-F\left(\alpha_{m}\right) \\
\beta_{1}^{n-m} P\left(\beta_{1}\right)-F\left(\beta_{1}\right) \\
\vdots \\
\beta_{n-m}^{n-m} P\left(\beta_{n-m}\right)-F\left(\beta_{n-m}\right)
\end{array}\right] .} \tag{9}
\end{align*}
$$

The results developed in this note also apply to multiinput multioutput systems using unity-rank output feedback matrices.

Example: For the system

$$
\dot{x}=\left[\begin{array}{rrr}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & -7
\end{array}\right] x+\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right] u ; \quad y=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right] x
$$

find the output feedback vector required to place two poles at $-1,-2$ and determine the residual characteristic polynomial.

In this example, we have

$$
\begin{array}{lc}
w(s)=C \operatorname{adj}(s I-A) b=\left[\begin{array}{c}
1 \\
s
\end{array}\right] ; & F(s)=|s I-A|=s^{3}+7 s^{2} \\
P(s)=(s+1)(s+2)=s^{2}+3 s+2 ; & Q(s)=s+a_{1}
\end{array}
$$

## Hence,

$$
\begin{aligned}
& M=\left[\begin{array}{ccc}
w(-1) & w(-2) & w(\beta) \\
0 & 0 & -P(\beta)
\end{array}\right]=\left[\begin{array}{rrc}
1 & 1 & 1 \\
-1 & -2 & \beta \\
0 & 0 & -\beta^{2}-3 \beta-2
\end{array}\right] \\
& d=[-F(-1),-F(-2), \beta P(\beta)-F(\beta)]=\left[-6,-20,-4 \beta^{2}+2 \beta\right]
\end{aligned}
$$

Choosing $\beta=0$ arbitrarily, we obtain

$$
\left[k_{1}, k_{2}, a_{1}\right]=d M^{-1}=[-6,-20,0]\left[\begin{array}{rrr}
1 & 1 & 1 \\
-1 & -2 & 0 \\
0 & 0 & -2
\end{array}\right]^{-1}=[8,14,4] .
$$

Hence, the required feedback vector is $k=[8,14]$ and the residual characteristic polynomial is $s+4$; i.e., the unassigned pole has moved to $s=-4$. It can readily be shown that the above solution is independent of $\beta$.

## Acknowledgment

The private communication with Dr. P. Murdoch, Brunel University, England, is gratefully acknowledged.

## References

[1] A. Jameson, "Design of single-input systems for specified roots using output feedback," IEEE Trans. Automat. Contr, vol. AC-15, pp. 345-348, 1970.
[2] E. J. Davison, "On pole assignment in linear systems with incomplete state feedback," IEEE Trans. Automat. Contr, vol. AC-15, pp. 348-351, 1970.
[3] B. Sridhar and D. P. Lindorff, "Pole-placement with constant gain output feedback," Jnt. J. Contr., vol. 18, no. 5, pp, 993-1003, 1973.
[4] F. Fallside and H. Seraji, "Pole-shifting procedure for multivariable systems using output feedback," Proc. Inst. Elec. Eng., vol. 118, no. 11, pp. 1648-1654, 1971.
[5] H. Seraji, "Direct method for pole assignment in single-input systems with output feedback," Electron. Lett, vol. 10, no. 12, pp. 243-244, 1974.
[6] -, "Unattainability of certain pole positions in single-input systems with output feedback," Int. J. Contr., vol. 22, no. 1, pp. 119-123, 1975.

## The Pole Shifting Theorem Revisited

## MICHAEL HEYMANN

## Abstract-It is shown that the pole shifting theorem can be regarded as a natural consequence of the uniqueness of solution to a certain discretetime time-optimal-control problem.

The purpose of the present communication is chiefly pedagogical. We consider the problem of pole shifting by state feedback in the single-input discrete-time system

$$
\begin{equation*}
x_{k+1}=A x_{k}+b u_{k} \quad(k=1,2,3, \cdots) \tag{1}
\end{equation*}
$$

where $x \in X:=K^{n}$ and $u \in U:=K^{1}, K$ being an arbitrary field. (The matrices $A$ and $b$ are $n \times n$ and $n \times 1 K$-matrices, respectively.) The characteristic polynomial of $A$, denoted $\psi_{A}(\lambda)$, is given by $\psi_{A}(\lambda)=\lambda^{n}+$ $\alpha_{1} \lambda^{n-1}+\cdots+\alpha_{n-1} \lambda+\alpha_{n}$.

We concern ourselves with the following well-known theorem.
Theorem 1: Assume that $(A, b)$ is reachable (i.e., that $Q:=$ $\left[b, A b, \cdots, A^{n-1} b\right]$ is nonsingular). Then for any polynomial $\phi(\lambda)=\lambda^{n}+$ $\beta_{1} \lambda^{n-1}+\cdots+\beta_{n-1} \lambda+\beta_{n}$ of degree $n$ with coefficients in $K$, there exists a $1 \times n K$-matrix $f$ such that $\phi(\lambda)$ is the characteristic polynomial of $A+b f$. Moreover, the matrix $f$ for which the above holds is unique.

Theorem 1 was first proved by Rissanen [1] in 1960 using what is now commonly called the "control canonical form." His proof of Theorem 1 relies on the fact that, as a consequence of reachability, the set of vectors $\left\{v_{1}, \cdots, v_{n}\right\}$, where $v_{n}=b$ and $v_{i}=A v_{i+1}+\alpha_{n-i} b, i=1, \cdots, n-1$, forms a basis for $K^{n}$. In this basis the system (1) is in control canonical form in which the pole shifting property becomes visually apparent. Theorem 1 was later discussed by Bass [2] who gave a somewhat different computational method for $f$ (see also Bass and Gura [3D. More recently, Ackermann [4] showed that $f$ can be expressed by the formula $f=$ $-e_{n} Q^{-1} \phi(A)$ where $e_{n}=(0,0, \cdots, 0,1)$, which has the advantage that, in contrast to the previously known methods, it does not require explicit knowledge of the characteristic polynomial of the open-loop matrix $A$.

While the above approaches to the proof of Theorem 1 (and to the computation of $f$ ) differ in detail, they share the direct reliance on the (technical) fact that $Q$ is nonsingular. Thus, the dependence of the (closed-loop) pole shifting property on the (open-loop) reachability property is commonly understood only circumstantially through the nonsingularity of $Q$, but not through any direct control theoretic insight.

In this communication we will show that Theorem 1 can be derived directly as a consequence of reachability and its "open-loop" consequences. In particular, it will be shown that the theorem is a natural consequence of the uniqueness of solution to the following discrete-time time-optimal-control problem which we denote by $(\mathbb{P})$ :

[^0]( P ) Assume that $(A, b)$ is reachable, and find a control sequence $\left\{u_{k}\right\}$ that transfers the initial state $x_{1}=b$ in the system (1) to the origin in a minimal number of steps.

As an immediate consequence of the reachability condition of $(A, b)$ and of the Cayley-Hamilton theorem we obtain the following.
Proposition 1: The time-optimal-control problem ( $\mathbb{P}$ ) has a unique solution with the following properties.
a) The minimal time is $l=n$ (that is, $x_{n+1}$ is the first state that can be zeroed).
b) The unique control sequence that solves (P) is given by $u_{i}=\alpha_{i}$, $i=1, \cdots, n$.
c) The optimal state sequence $\left\{x_{k}\right\}_{k=1}^{n}$ obtained by the controls of $b$ ) as

$$
\left\{\begin{array}{l}
x_{1}=b  \tag{2}\\
x_{k+1}=A x_{k}+b \alpha_{k} ; \quad k=1, \cdots, n-1
\end{array}\right.
$$

forms a basis for $K^{n}$ and is the unique state sequence through which $x_{1}=b$ can be steered to the origin in $n$ steps.

We consider now the problem (P) for a "feedback associate" of system (1), that is, for the system

$$
\begin{equation*}
x_{k+1}=\hat{A} x_{k}+b v_{k} ; \quad(k=1,2, \cdots) \tag{3}
\end{equation*}
$$

where $\hat{A}=A+b f$ with $f$ being a $1 \times n K$-matrix. The reachability of (1) obviously implies the reachability of (3) for every $f$.

Suppose first that $f$ is fixed and apply Proposition 1 to the system (3). Part b) of the proposition implies that the optimal control sequence $\left\{v_{i}\right\}_{i=1}^{n}$ is given by $v_{i}=\beta_{i}$ where the $\beta_{i}$ are the coefficients of the characteristic polynomial $\psi_{\hat{A}}(\lambda)$ of $\hat{A}$ :

$$
\psi_{\hat{A}}(\lambda)=\lambda^{n}+\beta_{1} \lambda^{n-1}+\cdots+\beta_{n-1} \lambda+\beta_{n} .
$$

The trajectories of the systems (1) and (3) can be equated by relating their controls through

$$
\begin{equation*}
u_{k}=v_{k}+f x_{k} ; \quad(k=1,2,3, \cdots) . \tag{4}
\end{equation*}
$$

Thus, the (unique) minimizing state sequence $\left\{x_{k}\right\}$ through which $x_{1}=b$ can be driven to the origin in $n$ steps is the same whether we employ the system (1) or (3). In other words, the state sequence in (2) is a "feedback invariant," that is, it is the same for every feedback associate of (1). From this latter fact and from (4), it follows that the coefficients $\beta_{i}$ are related to the $\alpha_{i}$ through $f$ by

$$
\begin{equation*}
\beta_{i}=\alpha_{i}+f x_{i} ; \quad(i=1,2, \cdots, n) \tag{5}
\end{equation*}
$$

We turn now to the converse problem.
Proof of Theorem 1: Let $\phi(\lambda)=\lambda^{n}+\beta_{1} \lambda^{n-1}+\cdots+\beta_{n-1} \lambda+\beta_{n}$ be any polynomial of degree $n$ with coefficients in $K$. We wish to find $f$ such that $\hat{A}=A+b f$ has $\phi(\lambda)$ as its characteristic polynomial. If such an $\hat{A}$ exists, then the state sequence $\left\{x_{k}\right\}_{k=1}^{n}$ of (2) is optimal also for the pair ( $\hat{A}, b$ ). Moreover, Proposition 1 [applied to the system (3)] implies that the sequence $v_{i}=\beta_{i}, i=1,2, \cdots, n$ must be the unique minimizing control sequence of problem ( $\mathbb{P}$ ) so that, as before, (5) must hold. That $\hat{A}$ indeed exists as required follows then from the fact that the optimal state sequence $\left\{x_{i}\right\}$ of (2) forms a basis for $K^{n}$ (part c) of Proposition 1) so that (5) has a unique solution $f$ for every set $\left\{\beta_{i}\right\}$.

From the above proof of the pole shifting theorem, it is apparent that the theorem can be regarded as a consequence of the uniqueness of the solution to the problem ( $\mathbb{P}$ ) and the "feedback invariance" of the state sequence (2). Also, a crucial fact on which the pole shifting theorem hinges is that the sequence (2) forms a basis for $K^{n}$. A similar point of view was also taken in a recent note by Hautus [5] where the so-called "Heymann Lemma," which extends the pole shifting theorem to multiinput reachable systems, is reproved.

The preceding discussion applies also when the reachability of $(A, b)$ is not satisfied. In that case, let $\phi_{(A, b)}(\lambda)$ be the minimal polynomial of $b$ (relative to $A$ ) (see, e.g., $\left[6\right.$, p. 176]). Then $\phi_{(A, b)}(A)$ is a factor of the
characteristic polynomial $\psi_{A}(\lambda)$ of $A$ and we can write $\psi_{A}(\lambda)=\xi(\lambda)$. $\phi_{(A, b)}(\lambda)$ for some polynomial $\xi(\lambda)$. It is then easily verified that $\xi(\lambda)$ is invariant under feedback, whereas $\phi_{(A, b)}(\lambda)$ can be arbitrarily changed by selection of $f$.

## References

[1] J. Rissanen, "Control system synthesis by analogue computer based on the 'generalized Linear feedback' concept", presented at the Int Seminar on Analog Computation Applied to the Study of Chemical Processes, Brussels, Belgium, 1960.
[2] R. W. Bass, "Control synthesis and optimization," NASA Langley Res. Center, Aug. 1961.
[3] R. W. Bass and I. Gura, "High order design via state space consideration," in Proc. 1965 Joint Auromat. Comtr. Conf., Troy, NY, pp. 311-318.
[4] J. Ackermann, "Der Entwurf linearer Regelungssysteme im Zustandsraum," Regelungstechnik, vol. 20, pp. 297-300, 1972.
[5] M. L. J. Hautus, "A simple proof of Heymann's lemma," IEEE Trans. Automar. Contr., vol. AC-22, pp. 885-886, 1977.
[6] F. R. Gantmacher, The Theory of Matrices, vol. 1. New York: Chelsea, 1959.

## On the Eigenvalue-Eigenvector Method for Solution of the Stationary Discrete Matrix Riccati Equation

## MICHAEL L. MICHELSEN

Abstract-The purpose of this correspondence is to point out that certain numerical problems encountered in the solution of the stationary discrete matrix Riccati equation by the eigenvalue-eigenvector method of Vaughan [1] can be avoided by a simple reformulation.

The positive definite solution matrix $P$ of the stationary discrete matrix Riccati equation

$$
\begin{equation*}
P=\Phi^{T}\left[P^{-1}+R\right]^{-1} \Phi+Q \tag{1}
\end{equation*}
$$

may be, as shown by Vaughan, found from the eigenvectors of the matrix

$$
k_{c}=\left[\begin{array}{rr}
\Phi^{-1} & \Phi^{-1} R  \tag{2}\\
Q \Phi^{-1} & \Phi^{T}+Q \Phi^{-1} R
\end{array}\right]
$$

The eigenvalues of $\kappa_{c}$ multiply pairwise to 1 , and $\kappa_{c}$ may be factorized into

$$
\kappa_{c}=\left[\begin{array}{ll}
W_{11} & W_{12}  \tag{3}\\
W_{21} & W_{22}
\end{array}\right]\left[\begin{array}{cc}
\Lambda & 0 \\
0 & \Lambda^{-1}
\end{array}\right]\left[\begin{array}{ll}
W_{11} & W_{12} \\
W_{21} & W_{22}
\end{array}\right]^{-1}
$$

where $\Lambda$ is diagonal with $\left|\Lambda_{i i}\right|>1$.
$P$ is subsequently found from the set of eigenvectors corresponding to eigenvalues of magnitude $>1$ :

$$
\begin{equation*}
P=W_{21} W_{11}^{-1} \tag{4}
\end{equation*}
$$

If the system matrix $\Phi$ has eigenvalues close to $0, \kappa_{c}$ will be severely ill-conditioned, and numerical accuracy is in particular lost in the evaluation of the term $\Phi^{T}+Q \Phi^{-1} R$ as small elements of $\Phi^{T}$ are added to large elements of $Q \Phi^{-1} R$.

This difficulty is avoided as follows.
The eigenvectors of the matrix

$$
\begin{equation*}
\kappa_{c}^{*}=\left(\kappa_{c}+I\right)^{-1}\left(\kappa_{c}-I\right) \tag{5}
\end{equation*}
$$

are identical to those of $\kappa_{c}$, and the eigenvalues of $\kappa_{c}^{*}$ occur pairwise with opposite signs. Those with positive real parts correspond to eigenvalues of $\kappa_{c}$ with magnitudes larger than 1.

Manuscript received June 3, 1978.
The author is with the Department of Chemical Engineering. Technical University of Denmark, Iyngby, Denmark.


[^0]:    Manuscript received March 28, 1978; revised August 26, 1978.
    The author is with the Department of Electrical Engineering, Technion-Israel Institute of Technology, Haifa, Israel.

